Corrigendum to the Elliptical Lorenz Curve by Villaseñor and Arnold (Journal of Econometrics 1989)

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Corrigendum to "Elliptical Lorenz Curves" [J. Econom. 40 (1989) 327 - 338], *Journal of Econometrics 174,* 2013, p.44

Abstract

The triparametric elliptical Lorenz Curve proposed by Villaseñor and Arnold in their Journal of Econometrics article in 1989 has proved very valuable for inequality researchers. Unfortunately, the paper contains a mistake in the conditions describing the parameter space, which numerous other papers have copied since. This note aims to correct the mistake in order to avoid confusion.

1 Corrigendum

In their seminal Journal of Econometrics article, Villaseñor and Arnold (1989) propose the following triparametric elliptical Lorenz Curve

$$L(\pi, a, b, d) = \frac{1}{2}\left[-(b\pi + e) - \sqrt{\alpha\pi^2 + \beta\pi + e^2}\right] \tag{1}$$

where $e = -(a + b + d + 1)$, $\alpha = b^2 - 4a < 0$ (so that the curve is an ellipse rather than a parabola or hyperbola) and $\beta = 2be - 4d$.
Following Pakes (1981), a function $L(\pi)$, continuous on $[0, 1]$ and with second derivative $L''(\pi)$, is a Lorenz Curve (LC) if and only if

$$L(0) = 0,\ L(1) = 1,\ L'(0^+) \geq 0,\ L''(\pi) \geq 0 \text{ in (0,1)} \tag{2}$$

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In order for the proposed form (1) to fulfill these four criteria and be called a Lorenz Curve, Villaseñor and Arnold (1989) state the following four necessary and sufficient parameter conditions in their Lemma 1:

$$\alpha = b^2 - 4a < 0 \tag{3}$$
$$a + b + d + 1 > 0, \tag{4}$$
$$d \geq 0 \tag{5}$$
$$a + d - 1 \leq 0 \tag{6}$$

However, the last of these conditions is wrong: It has to read

$$a + d - 1 \geq 0 \tag{7}$$

Interestingly, although the elliptical Lorenz Curve has proved very valuable in theoretical research and empirical studies, with a vast amount of papers citing Villaseñor and Arnold (1989), no one seems to have pointed out this mistake explicitly. Instead, numerous papers have copied the incorrect condition, such as Schader and Schmid (1994) and Sarabia (2008).

The sign reversal can have far-reaching consequences: It is crucial for the Lorenz Curve criterion $L(1) = 1$ to be satisfied. For example, the parameter combination $a = 0.2$, $b = 0$, $d = 0.1$, which is part of the incorrect parameter space (3), (4), (5), (6) implies a curve going through $L(1) = 0.3$ rather than $L(1) = 1$. By condition (7), it would not be a valid combination. One can prove that condition (7) is directly related to the requirement $L(1) = 1$. In the Appendix, I show and confirm how (3), (4), (5) and (7) follow from the Lorenz Curve requirements in (2).

For theoretical research in the inequality field, working with the correct parameter space for Lorenz Curves is vital. It might be less important for applied work fitting the elliptical LC (1) to empirical data points which always fulfill $L(1) = 1$. But even there, incorporating an incorrect restriction into an estimation procedure can lead to non-convergence and other problems. Given the wide use of the elliptical Lorenz Curve and the proliferation of the incorrect condition (6) in the literature based on the Journal of Econometrics article, this corrigendum aims to draw attention to the mistake in order to avoid confusion.

References

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2 Appendix

It will be shown that the four (correct) conditions (3), (4), (5) and (7) are both necessary and sufficient for the function (1) to fulfill the four Lorenz Curve requirements stated in (2). In fact, (3) defines the curve as an ellipse rather than a parabola or hyperbola, so only the remaining three conditions are to be shown.

- The criterion $L(0) = 0$ is equivalent to

$$\begin{aligned} \frac{1}{2}\Big[-(b \cdot 0 + e) - \sqrt{\alpha \cdot 0^2 + \beta \cdot 0 + e^2}\Big] &= 0 && (8) \\ -e - \sqrt{e^2} &= 0 && (9) \\ e &\leq 0 && (10) \end{aligned}$$

- The criterion $L(1) = 1$ is equivalent to

$$\frac{1}{2}\Big[-(b \cdot 1 + e) - \sqrt{\alpha \cdot 1^2 + \beta \cdot 1 + e^2}\Big] = 1 \qquad (11)$$

$$-(b + (-a - b - d - 1)) - \sqrt{b^2 - 4a + 2b(-a - b - d - 1) - 4d + (-a - b - d - 1)^2} = 2$$

$$-a - d - 1 +$$

$$\sqrt{b^2 - 4a - 2ab - 2b^2 - 2bd - 2b - 4d + a^{+}b^2 + d^2 + 1 + 2ab + 2ad + 2bd + 2a + 2b + 2d}$$

$$= -2$$

This can be simplified to

$$\begin{aligned} \sqrt{a^2 + 2ad + d^2 + 1 - 2a - 2d} &= a + d - 1 && (12) \\ \sqrt{(a + d)^2 - 2(a + d) + 1} &= a + d - 1 && (13) \\ \sqrt{(a + d - 1)^2} &= a + d - 1 && (14) \\ a + d - 1 &\geq 0 && (15) \end{aligned}$$

- The criterion $L'(0^+) \geq 0$ is equivalent to

$$\begin{aligned} \frac{1}{2}\Big[-b - \frac{1}{2}\frac{1}{\sqrt{\alpha\pi^2 + \beta\pi + e^2}}(2\alpha\pi + \beta)\Big] &\geq 0 && (16) \\ \frac{1}{2}(2\alpha\pi + \beta) &\leq -b \cdot \sqrt{\alpha\pi^2 + \beta\pi + e^2} && (17) \\ \frac{1}{2}\Big(2(b^2 - 4a)\pi + 2be - 4d\Big) &\leq -b \cdot \sqrt{\alpha\pi^2 + \beta\pi + e^2} && (18) \\ (b^2 - 4a)\pi + be - 2d &\leq -b \cdot \sqrt{\alpha\pi^2 + \beta\pi + e^2} && (19) \end{aligned}$$

This is equivalent to

$$d \geq 0 \tag{20}$$

In fact, the first term is negative because $\alpha = b^2 - 4a < 0$ by (3) and the second term has the sign of $-b$ as $e < 0$. For $\pi \to 0$, the first term goes to zero and the second term equals the RHS of the equation, as then the square root is $-e$. The term $-2d$ makes the difference and ensures that the equation is fulfilled if and only if $d \geq 0$.

- The criterion $L''(\pi) \geq 0$ is equivalent to

$$\begin{aligned} \frac{1}{2} \frac{\frac{1}{4}\beta^2 - \alpha e^2}{\sqrt{(\alpha\pi^2 + \beta\pi + e^2)^3}} &\geq 0 && (21) \\ \frac{1}{4}\beta^2 - \alpha e^2 &\geq 0 && (22) \end{aligned}$$

For all elliptical LCs with (3) $\alpha = b^2 - 4a < 0$, this condition is fulfilled.

Taken together we have the four conditions

$$\begin{aligned} \alpha = b^2 - 4a &< 0 && (23) \\ e = -(a + b + d + 1) &< 0, && (24) \\ d &\geq 0 && (25) \\ a + d - 1 &\geq 0 && (26) \end{aligned}$$

which equal (3), (4), (5) and (7).[1]

[1]From (8), $e = 0$ might seem an option as well. $e = 0$ does fulfill $L(0) = 0$ but clashes with the other conditions. One can show that there is no combination of a, b and d with $e = -(a + b + d + 1) = 0$ which satisfies $d \geq 0$ and $b^2 - 4a < 0$. Indeed, $b^2 - 4a = b^2 - 4 \cdot (-1 - b - d) = (b + 2)^2 + 4d$ can never be smaller than zero for $d \geq 0$. So $e = 0$ is ruled out and the parameter requirement from (8) can be simplified to $e < 0$.